2.9 Continuity

(a very preliminary version, I'd like to make it more constructive)

Let us assume there is a function f and we want to calculate $f(x)$. In "real life" situations we almost never know the number x exactly, and the following question becomes important: "how much accuracy do we need in x in order to get a certain accuracy in $f(x)$?" Or, to put it more precisely:

How many accurate decimal places in x do we
need to get n accurate decimal places in $f(x)$? (?n)

Here are a few examples.

1. $f(x) = 100x + 7$, then by taking $n + 2$ accurate decimal places in x we get n accurate decimal places in $f(x)$, no matter what x is.
2. $f(x) = x^2$ and assume $|2x| < 10^k$, then by taking $n + k$ accurate decimal places in x we get n accurate decimal places in $f(x)$.
3. $f(x) = \sqrt(x)$, $x = 0$, then we need $2n$ accurate decimal places in x to get n accurate decimal places in $f(x)$, and it will work for $x > 0$ as well.
4. $f(x) = 1/x$ and $|x| > 10^{-k}$, then we can get n accurate decimal places in $f(x)$ by taking $n + 2k$ accurate decimal places in x.
5. $f(x) = sin(x)$, then we can get n accurate places in $f(x)$ by taking n accurate places in x.

Exercises

1) Verify the claims in these examples. (Hint: use the fact that the chord is shorter than the corresponding arc to treat the example 5.) 2) Generilize example 2 to $f(x) = x^m$ and example 3 to $x^{1/m}$.

These examples suggest that as long as x stays away from the "bad" values (such as $x = 0$ for $f(x) = 1/x$) and from infinity (which means that there is an estimate of the form $|x| < A$, like in example 2), we can answer the question (?n) in a satisfactory manner. In othes words, given n, we can, by taking enough (but still a finite number) of accurate decimal places in x get n accurate decimal places in $f(x)$.

Such functions are called continuous. From the computational point of view, we can only deal with continuous functions because most computations are done with finite accuracy. There are even some mathematicians that refuse do deal with functions that are not continuous, they say that only continuous functions are well defined. We will discuss this and related topics more carefully in Chapter 5.

Actually there are two brands of continuity.

If we fix x first and then worry about the question (?n), we get continuity at this particular x.

If we consider the whole range of values for x and then worry about the question (?n), we get the uniform continuity (for this particular range of values of x).

This brand of continuity is more important for practical purposes.

There is a theorem by E. Heine (1872) that says that if a function f is continuous at every x such that $a \leq x \leq b$, then f is uniformly continuous on the whole closed interval $[a, b]$ (which is the set of numbers x such that $a \leq x \leq b$).

This theorem becomes wrong if we replace one of the $\leq$ signs with the $<$ sign. We can understand why by inspecting in more detail the function $1/x$ from example 4. It is continuous at every x of the interval $(0, 1]$, but not uniformly continuous on this interval (Exercise: Check it).

We will mostly deal with continuous functions on closed intervals and by continuity will mean the uniform continuity. Continuity at a given point will be less important. In fact the whole notion of a given point becomes problematic when we deal only with the finite accuracy approximations, but it is still handy for the theory.

Continuos functions are rather reasonable, in particular, *continuous functions don't jump,* in other words, *if f is a continuous function defined on an interval (a, b) and $f(x) = 0$ for all $x \neq c$ then $f(c) = 0$ too.*

Indeed, there is $d \neq c$ such that $a < d < b$ and $d - c$ is as small as we want, but $f(d) = 0$, therefore $f(c) \approx 0$ with any accuracy we want, therefore we must have $f(c) = 0$.

The following properties of continuous functions are immediate.

1. A sum of two continuous functions is continous.
2. A constant multiple of a continuous function is continuous.

It follows that our approach to differentiation (see section 2.1) works for continuous functions, i.e. the rule that $f'(a)$ is $(f(x) - f(a))/(x - a)$ evaluated at $x = a$ defines $f'(a)$ unambiguously if the division is carried out in the class of continuous functions.

It follows from the observation that any 2 continuous functions g and h such that $(x - a)(g(x) - h(x)) = 0$ must be equal because they are equal for $x \neq a$ as well as for $x = a$ ($g - h$ can't jump).

Exercises

(This is an outline of another approach to continuity using the moduli of continuity, all functions are defined on a closed interval)

3) An increasing function that hits all its intermediate values is continuous.

4) The inverse of an increasing function is continous.

5) Bolzano theorem says that a continouous function defined on $[a, b]$ hits all the values between $f(a)$ and $f(b)$. Derive the following: an increasing function is continuous if and only if it hits all its intermediate values.

6) A continuous function that has an inverse must be monotonic (= increasing or decreasing). (Hint: Use Bolzano).

7) Assume that $|f(x)-f(a)| \le g(|x-a|)$ with increasing continuous g and $g(0)=0$. Then f is continuous at a.

8) Let $|f(x+h)-f(x)| \le g(|h|)$ for any x, with g as in the previous exercise. Then f is uniformly continuous.

Exercises 3, 7 and 8 can be taken as the definitions of continuity.