7 Improper integrals, infinite series, limits

Problem 7.1. Calculate the following integral

$$\int_0^4 \frac{dx}{\sqrt{x}}$$

This integral is called *improper* because the integrand $1/\sqrt{x}$ blows up at 0. Nevertheless, it doesn't blow up too badly, so the area under the graph is still finite.

Problem 7.2. Calculate

$$\int_1^\infty \frac{dx}{x^2}$$

This integral is improper because the upper limit is infinite, but the integrand falls off fast enough to keep the area under the graph finite. Can you see the connection between these two integrals?

Problem 7.3. Look at the infinitely nested square root:

$$\sqrt{6+\sqrt{6+\sqrt{6+...}}}$$

Can you figure out what it is equal to? Think about iterations that we discussed during the class on August 7.

Problem 7.4. **a)** Compare the infinite sum

$$1 + 1/4 + ... + 1/n^2 + ...$$

with the integral from 7.2 by representing this infinite sum as the total area of an infinite bunch of rectangles, one for each n, of length 1 and height $1/n^2$. Convince yourself that this infinite sum has a finite value.
b) Try to generalize to

$$1 + 1/2^p + ... + 1/n^p + ...$$

to derive *the p-test*: this infinite sum *converges*, i.e. the truncated sum

$$1 + 1/2^p + ... + 1/n^p$$

approaches some finite value as $n \to \infty$, if $p > 1$, otherwise it *divirges*, i.e. the truncated sum $\to \infty$ when $n \to \infty$.

Problem 7.5. Take a look at yet another improper integral:

$$\int_0^\infty \frac{sin(\pi x)}{x} dx = \lim_{a \to \infty} \int_0^a \frac{sin(\pi x)}{x} dx$$

and try to understand why it is finite, while the area under the graph of the integrand and over the x-axis and the area over the graph of the integrand and under the x-axis are both infinite.

Compare this integral to the infinte series

$$1 - 1/2 + 1/3 - 1/4 + ... + 1/(2n-1) - 1/(2n) + ...,$$

here the total sum is finite, while the sum of the odd terms and the sum of the even terms are both infinite.

Take a look at the additional readings and problems # 7 and section 2.8 of the lecture notes.