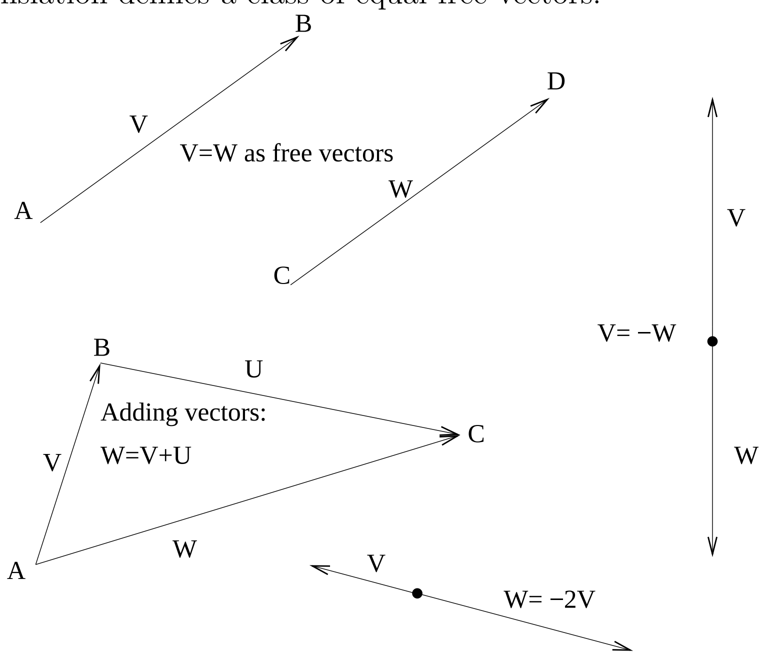
4 Trigs, log and exp

Problem 4.1. Vectors Vectors are just segments of straight lines, with one end considered the head and the other the tail. The head is marked with an arrow. For example, vector V has the head B and the tail A. Vector V "drags" point A to the new position B. We can move vector V to a new position W, keeping it parallel to itself. We consider W and V equal as *free* vectors, i.e. 2 free vectors are equal if they have the same length and the same direction. We can think of free vectors as *parallel translations* of the plane. Any free vector defines a parallel translation of the plane that drags each point in the direction of this vector the distance equal to its length. Any 2 equal free vectors define the same parallel translation and any parallel translation defines a class of equal free vectors.



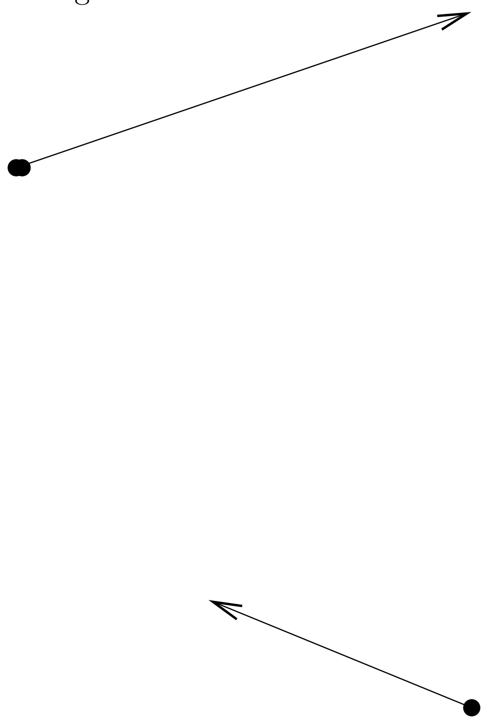
When we do 2 parallel translations by 2 free vectors U and V in a row, the total result is also a parallel translation by the free vector W which is called *their sum*, $W = V + U$.

a) Show that $U + V = V + U$

Vectors can be multiplied by numbers (see the figure), aV points in the same direction as V if a is positive and in the opposite direction if a is negative, the length of aV, $|aV| = |a||V|$.

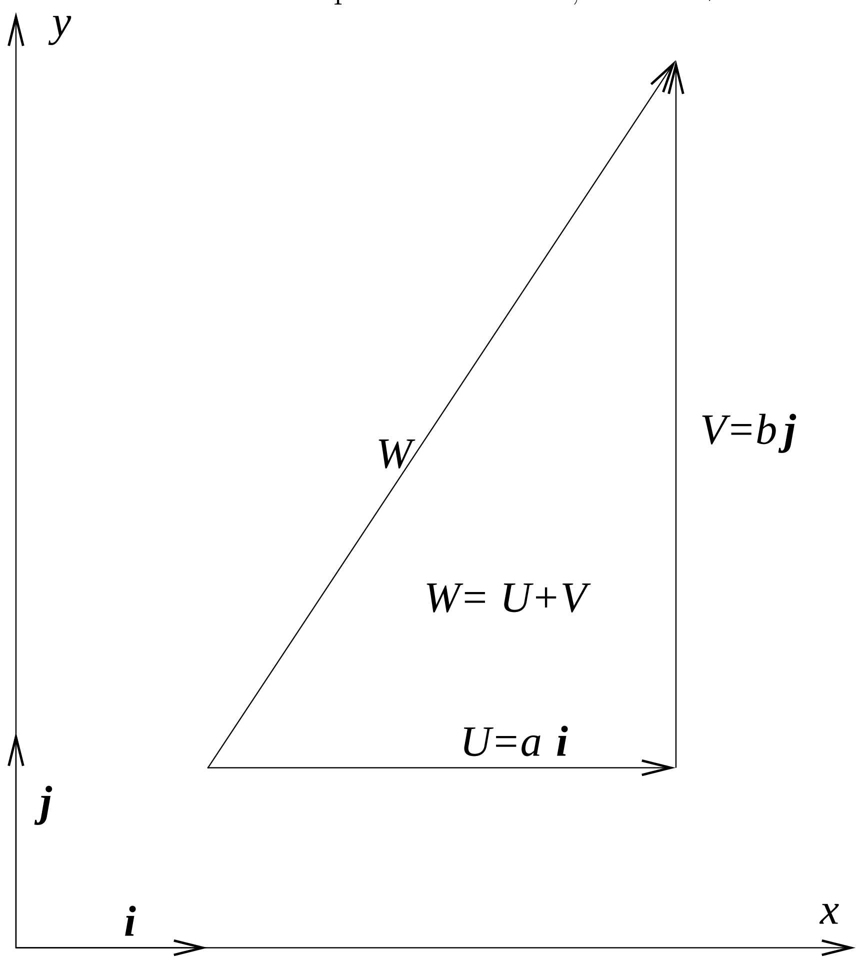
b) What about zero vector?

c) The positions and the velocities of 2 ships at some moment are shown on the figure.



If the ships keep their velocities constant, find the minimal distance between the ships (assume that the ocean is flat). See the file 2ships.pdf from additional readings for a stimulating discussion of this problem by George Polya.

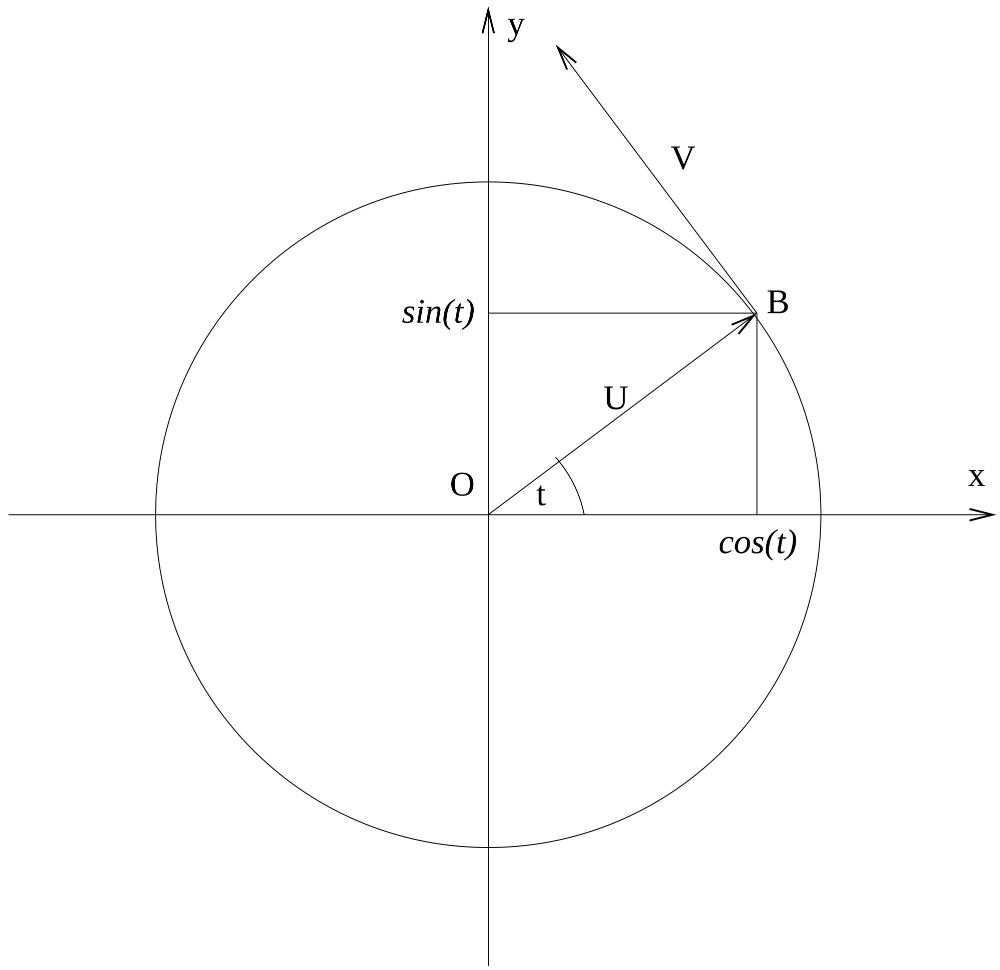
On the coordinate $x - y$ plane any vector W can be decomposed into its horizontal and vertical components U and V, $W = U + V$.



Here $\mathbf{i}$ and $\mathbf{j}$ are the horizontal and the vertical vectors of unit length.

Problem 4.2. Trig derivatives

a) sin' **and** cos' Now think about the point B that moves around the unit circle with the unit speed. Then the angle between the vector U and x-axis is equal to time t (see the diagram).



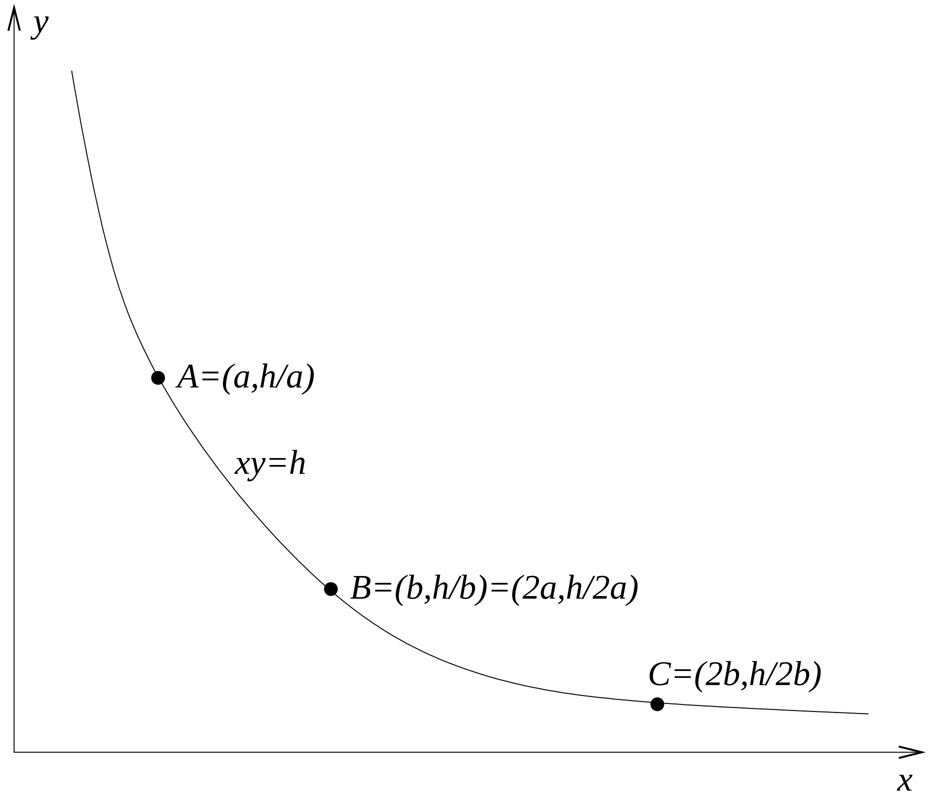
Vector V is the velocity of point B, $|U| = |V| = 1$. By thinking of $cos'(t)$ as the horizontal velocity of point B and of $sin'(t)$ as its vertical velocity, figure out sin' and cos'. See section 2.4 of the lecture notes if you are stuck.

b) Use differentiation rules to get tan' and cot'.

c) Get $arcsin'$ and $arccos'$ by implicit differentiation.

Problem 4.3. Hyperbolic rotations and logarithms It is clear from the previous problem that the trig functions are related to the rotations around the origin. These rotations preserve x^2+y^2 that is the square of the distance from the origin.

Similarly, logarithms are related to the *hyperbolic* rotations that preserve $4xy = (x+y)^2 - (x-y)^2$. Here is an example of hyperbolic rotation.



It moves any point A with coordinates $(a, h/a)$ to the point B with the coordinates $(b, h/b) = (2a/h/(2a))$. Both points A and B lie on the same hyperbola $xy = h$, i.e. our transformation drags the poins along the hyperbolas, while the ordinary rotations drag the points along the circles. Taking some other constant c instead of 2, we get other hyperbolic rotations.

Check that all these hyperbolic rotations preserve area, i.e. they transform any region on the plane into the regions of the same area.

Hyperbolic rotations are used in section 2.4 of the lecture notes to define the natural logarithm.

Problem 4.4. Power series

a) Write down the Maclaurin series for $exp(x)$, $1/(1-x)$, $sin(x)$, $cos(x)$ (see problem 3.4).

b) Derive that $e^{it} = cos(t) + isin(t)$, where $i = \sqrt{-1}$, by using the Maclaurin series.

c) Write down the Maclaurin series for $(1+x)^a$. This series was engraved on Newton's gravestone because it was one of his favorite results.

d) Derive the binomial formula from c) for $a = n$.