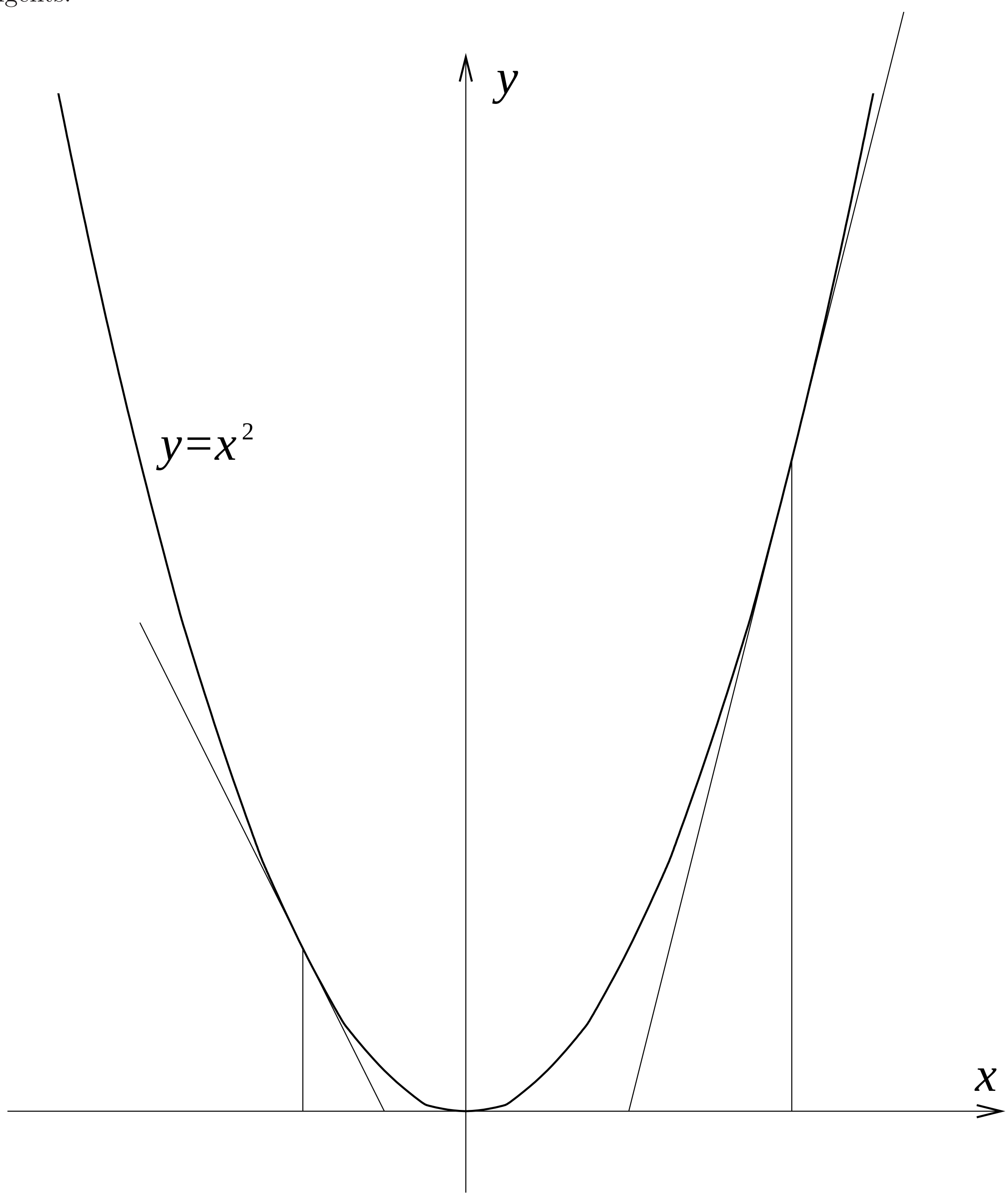
1 Areas, Sums and Tangents

1.1 Problems

Problem 1.1. Look at the diagram that shows a parabola and some of its tangents.

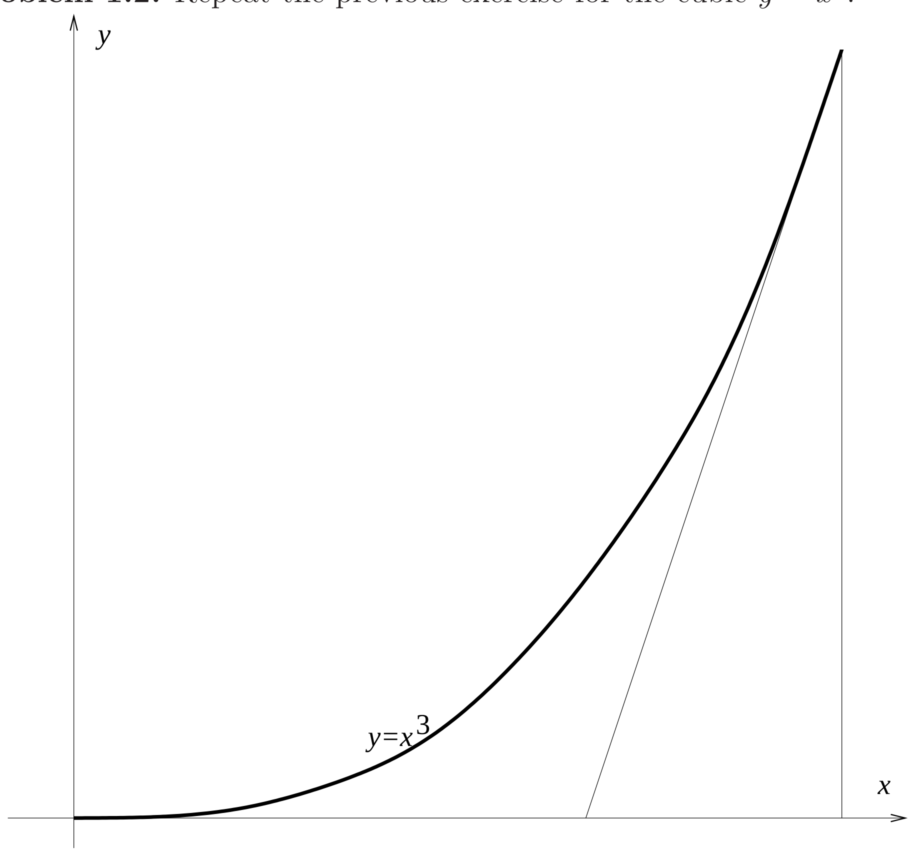


Do you see anything peculiar about the points of intersection of the tangents with the x-axis? Based on your observations, suggest a method of drawing the tangent at a given point (a, a^2) of this parabola, using a straight edge and a drawing compass.

Let $y = a^2 + b(x - a)$ be the equation of the tangent to $y = x^2$ at point (a, a^2). Express b in terms of a using the following reasoning. We know that the x-axis is tangent to $y = x^2$ at the origin $(0, 0)$ and $x = 0$ is a double root of the equation $x^2 = 0$. So it's reasonable to assume that $x = a$ is a double root of the equation $x^2 - a^2 - b(x - a) = 0$.

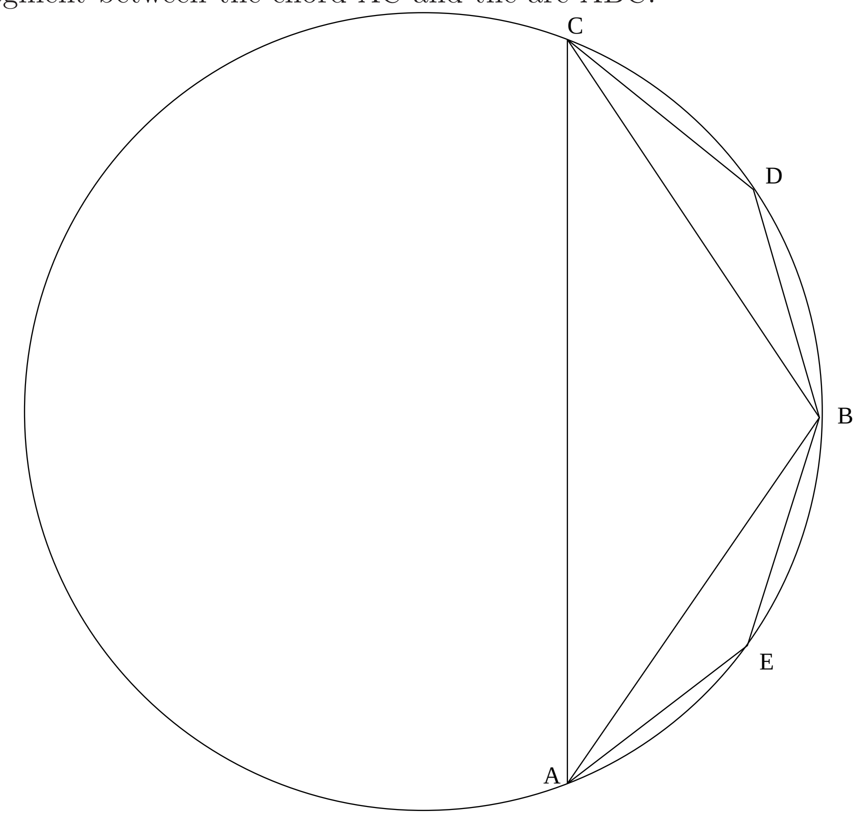
Using the formula for b in terms of a, justify your method for drawing the tangents to the parabola.

Problem 1.2. Repeat the previous exercise for the cubic $y = x^3$.



Problem 1.3. The area of a parabolic segment (after Archimedes).

a) Warmup: The area of a circular segment. First consider the circular segment between the chord AC and the arc ABC.

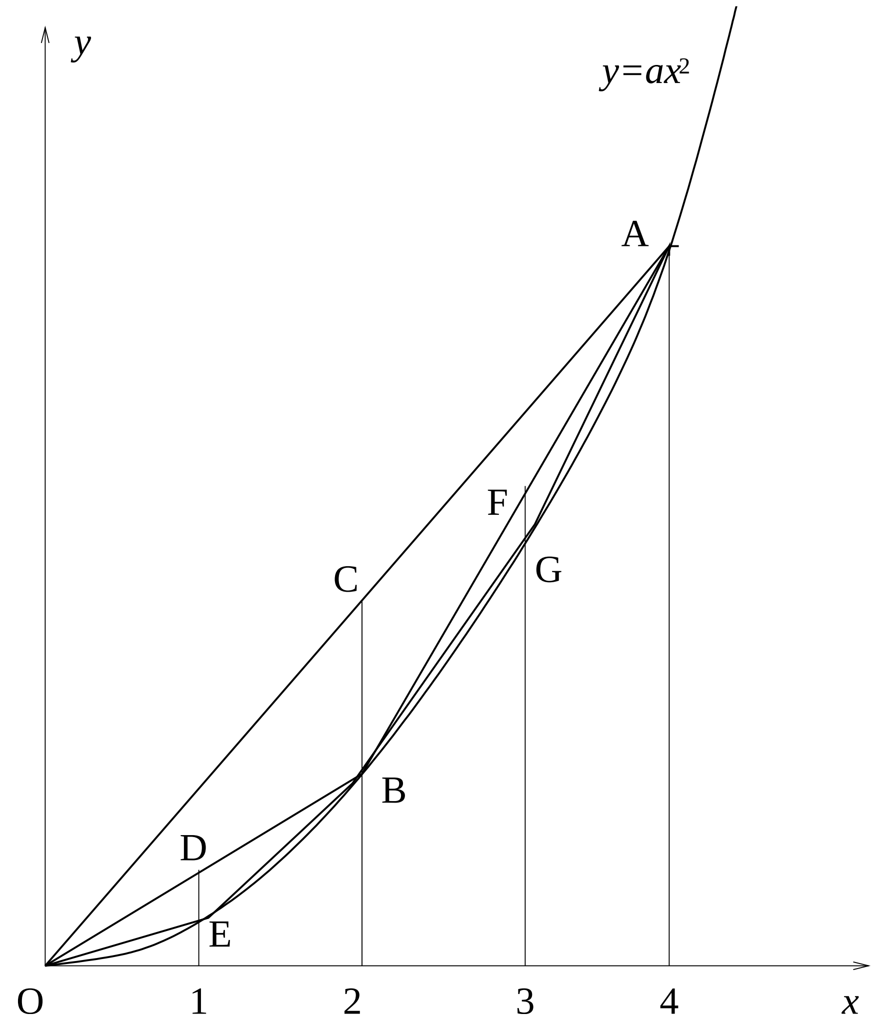


We can approximate the area of the segment ABC by the areas of polygons inscribed into it. This is the essence of the exhaustion method used by the Greeks.

First we observe that the area of the triangle ABC is more that a half of the area of the segment (B is the mid-point of the arc ABC). Try to prove it.

It follows that the area of the segment AEB + the area of the segment CDB is less that half of the area of the segment ABC. Now, bisecting both of the arcs AB and CB by E and D, we can see that the area of the polygon ACDBE is greater than 3/4 of the area of the segment ABC. We can continue bisecting the arcs and adding new triangles to the polygon. Show that after N bisections we get a polygon with area that is at least $1 - 2^{-N}$ the area of the segment ABC.

b)The area of a parabolic segment. Now we take up a parabolic segment.



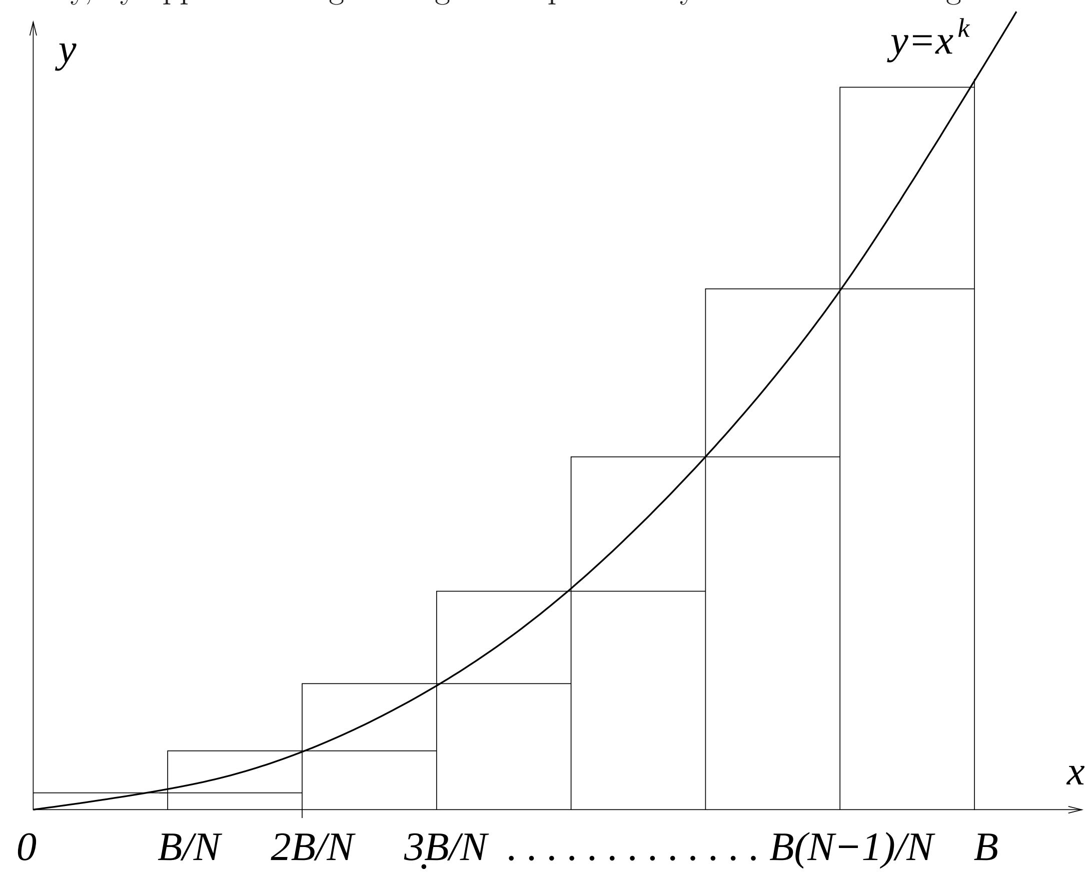
Show that the tangent to the parabola at point B is parallel to the chord OA and conclude that the area of the triangle ABO is greater than half of the area of the parabolic segment. Do the same for OEB and BGA.

Find an explicit explression for the area of the triangle OBA. Find an explicit explression for the area of the triangles OEB and BGA. Check that the area that we add after any subdivision is $1/4$ of the area added after the previous one. so the total area of the segment ABO is $1 + 1/4 + 1/16 + ...$ times the area of the triangle ABO. Now you can derive the explicit formula for the area of the segment ABO.

c)The sum of a geometric series Derive a formula for the sum $1 + R + R^2 + R^3 + ... + R^N$ and for the infinite sum $1 + R + R^2 + R^3 + ...$ when $|R| < 1$.

Problem 1.4. Sums of whole powers and the uniform grid approach to the area under $y = x^k$ for a whole k.

The method of used in the previous execise breaks down for $y = x^k$ with $k > 2$. Nevertheless, the area under $y = x^k$ for natural k can be done similarly, by approximating the region in question by a bunch of rectangles.



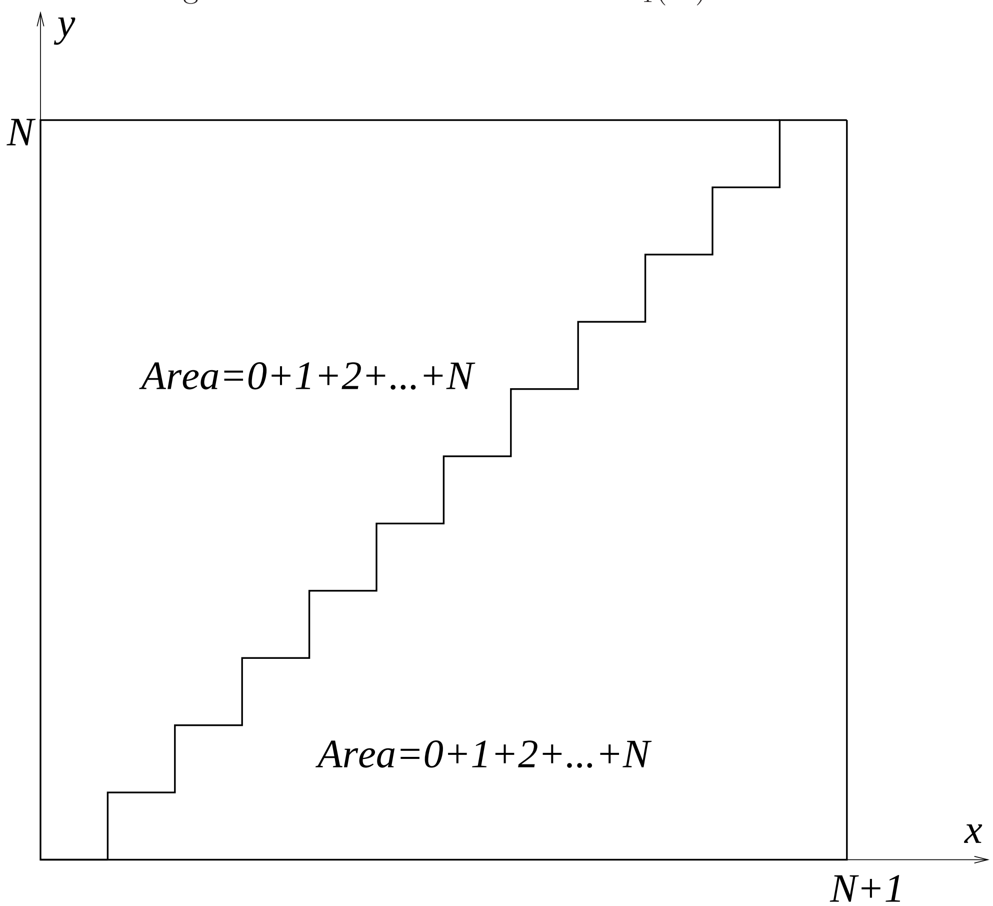
So here we have two staircase-like polygons, one sits inside the the area under our curve and the other contains this area. The areas of these polygonal regions approximate the area under the curve from above and from below, and for large N the approximating areas become close. So we get the area under our curve if we put $N = \infty$ in the formula for the area of one of the approximating polygonal regoins (no matter which one).

Now you can go ahead and calculate the areas of the approximating polygonal regions. In the process you will encounter the sums

$$S_k(N) = 1^k + 2^k + 3^k + ... + N^k$$

The case $k = 0$ is trivial, $k = 1$ is rather simple and was on the entrance test for this class, $k = 2$ is a bit trickier, it is explained in section 3.2 of G. Polya's Mathematical Discovery, and the derivation of the formulas for bigger k is given in sections 3.3 and 3.4, the whole chapter 3 is available on ftp://hssp06.mathfoolery.org/HSSP_scans/mathdiscovery.ch3.pdf

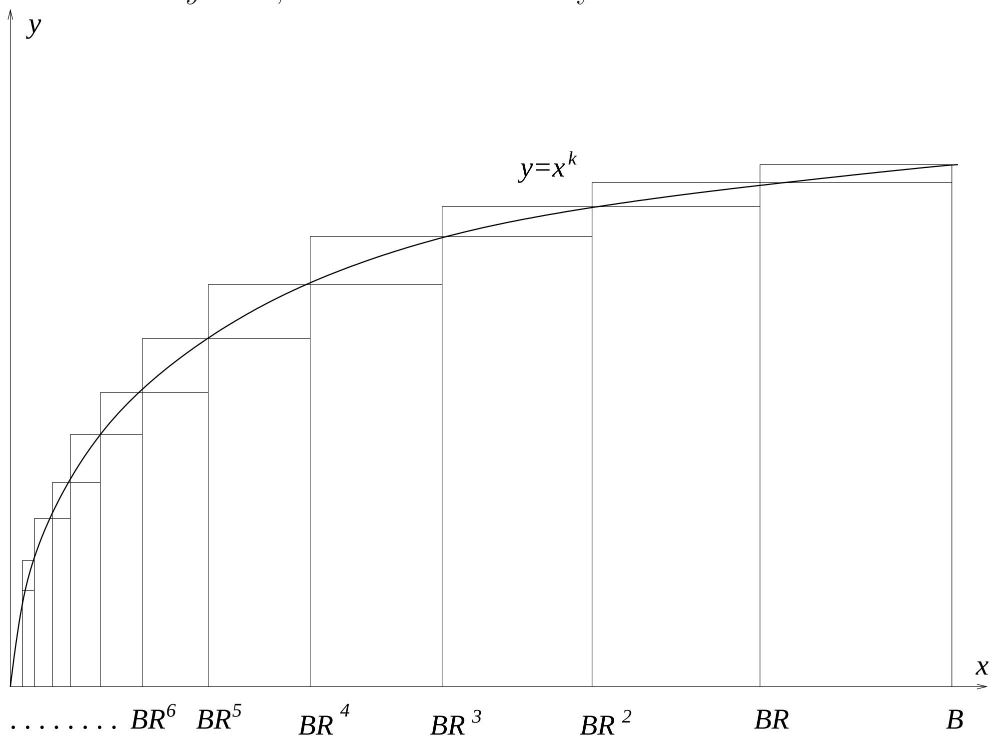
The next diagram illustrates the formula for $S_1(N)$.



So $S_1(N) = N(N+1)/2$ and we recover the area of a right triangle as the area under the curve $y = cx$. In the general case the area of the approximating polygonal regions will be of the form $P(N)/N^{k+1}$, where P is a polynomial of degree $k+1$, and the formula makes sense for $N = \infty$.

Problem 1.5. The area under the curve $y = x^k$ (after Fermat)

Following Fermat, you can use the geometric series to calcullate the area under the curve $y = x^k$, with k not necessarily a whole number.



Here $0 < R < 1$. that is close to 1. The region under the curve is contained inside the infinite staircase region and contains the other infinite staircase region. Express the area of both inner and outer staircase regions as sums of the geometric series with the denominator R. Convince yourself that the inner and the outer areas get close when R is close to 1, so we get the area under the curve when we take $R = 1$ in the formula for, say, the outer area. You will need some algebraic tricks to evaluate this expression, if you do it directly, you will get $0/0$. Consider first the case of k whole or a reciprocal of a whole. Check if the result for $k = 2$ agrees with two previous problems.

Making sense of $0/0$ is the essence of differentiation and calculating areas under curves is the essence of integration. This problem shows that differentiation and integration are intimately related.

Problem 1.6. Differentiating x^2, x^3 etc.

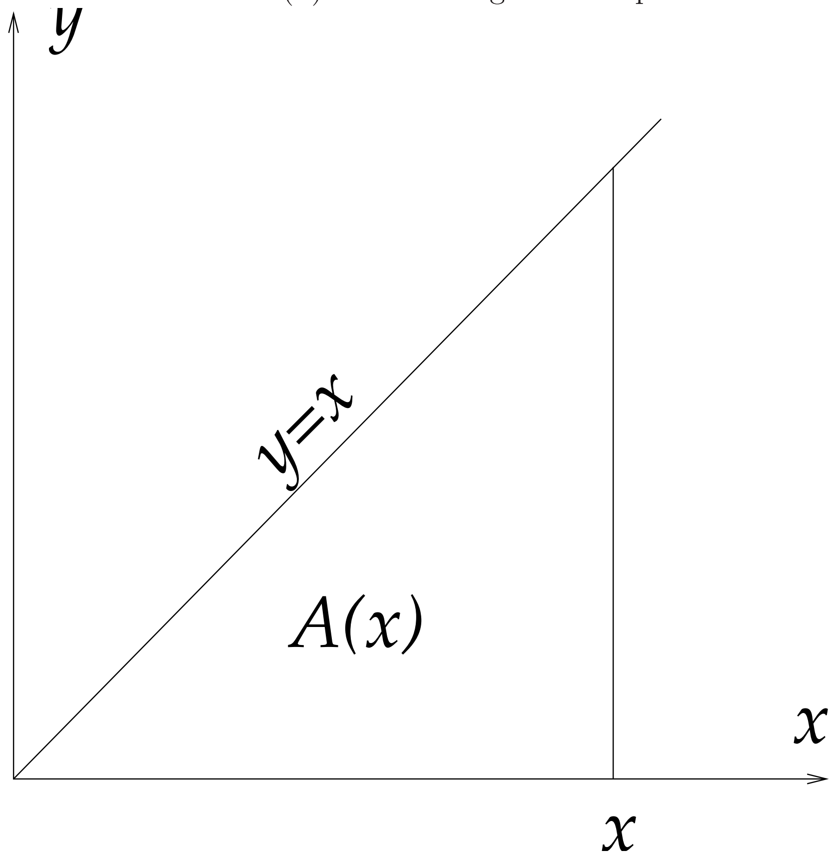
Can you make sense of $(x^2 - a^2)/(x - a)$ for $x = a$? Notice that if you just plug in $x = a$ into this formula, you will get $0/0$ which is undefined, so you have to be a bit clever here. Can you see any connection between this problem and problem 1.1? The answer to this problem is the number that depends on x, in other words, it is a function of x. This function is called *the derivative* of x^2 and is denoted by $(x^2)'$. The process of finding the derivative of a given function is called *differentiation.*

Repeat for x^3 instead of x^2, i.e. try to make sense of $(x^3 - a^3)/(x - a)$ for $x = a$. Can you see the connection with problem 1.2?

Try to generalize to x^n, this will be the formula for $(x^n)'$.

Problem 1.7. Newton-Leibniz by example.

Consider the area $A(x)$ of the triangle on the picture.



Differentiate $A(x)$. What is the connection between the two functions $A'(x)$ an x?

Repeat for $y = x^2$ instead of $y = x$ if you know the area under $y = x^2$ (see problem 1.4).

Try to generalize to $y = x^k$.